TN NO. 66T-5

# UNITED STATES NAVAL POSTGRADUATE SCHOOL

## DEPARTMENT OF AERONAUTICS

## TECHNICAL NOTE

NO. 66T-5

TEMPERATURE DISTRIBUTIONS IN THIN CIRCULAR DISKS WITH VARIABLE THICKNESS, THERMAL CONDUCTIVITY, AND HEAT TRANSFER COEFFICIENT (A DIRECT VARIATIONAL APPROACH USING RAYLEIGH RITZ APPROXIMATION)

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ABSTRACT

An approximate solution for the temperature distribution in a thin circular disk having variable thermal properties, variable thickness, and a variable surface heat transfer coefficient is presented. The method used was to cast the governing differential equation into the equivalent variational problem and solve using the direct method of Rayleigh Ritz. Agreement between the approximate solution and an exact solution for the case of a constant thickness disk with constant properties was excellent. A simple computer program which may be used to evaluate the temperature profiles is provided.

TABLE OF CONTENTS

## LIST OF FIGURES

## LIST OF TABLES

## LIST OF SYMBOLS

$k$ thermal conductivity

$T$ temperature

$Q_s$ heat sink term

$h$ heat transfer coefficient

$b$ disk thickness

$T$ ambient temperature

$R$ disk outside radius

$\theta$ non-dimensional temperature

$\eta$ non-dimensional radius

$\lambda$ non-dimensional property parameter

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Applications requiring knowledge of temperature distributions in disks are many; probably the most common being turbine rotors. Exact solution for the temperature distributions in disks with variable properties is at present impossible except for a few special cases, however approximate techniques may be used which allow solution of these problems. The very powerful variational calculus approach used in this report is one of these techniques. The excellent agreement between the approximate solution and an exact solution for a particular case encourages confidence in the analysis.

The primary motivation for this analysis was the thermal stress problem in turbine rotors which requires knowledge of the temperature distribution (for the uncoupled problem). Large temperature gradients may be present which result in large property variation. Flexibility in the present solution allows an iterative approach to the problem of determining a temperature distribution for this complex case. The temperature distribution may be determined for an arbitrary distribution of properties, then the new temperature distribution used to obtain a new set of property variation and so forth to the desired degree of convergence. It is expected that convergence would be quite rapid. A second intriguing application for those who enjoy treading upon thin ice would be to experimentally determine the temperature distribution in the disk (rotating or not)

then deduce the required heat transfer coefficient distribution to produce that temperature distribution. This approach would provide simple instrumentation for rotating disks with complex flow conditions existing on the faces and still give reasonable quantitative heat transfer information.

Formulation of the Problem.

Consider an axisymmetric thin circular disk of outside radius R and arbitrary thickness distribution b(r). The only requirement which we place is that the disk be thin enough to ignore temperature variations in the axial direction. This allows us to treat the problem as a normal heat conduction problem with a uniformly distributed heat sink. We may write the governing equation:

$$k\nabla^2 T - Q_s = 0$$

The heat sink term is expressed as a function of the heat transfer coefficient on the face of the disk.

$$Q_s = \frac{2h(T - T_\infty)}{b}$$

If the dimensionless variables $\theta, \eta$ are introduced where $T_\infty$ is the film, (or bulk) temperature for the heat transfer coefficient,

$$\theta = T - T_\infty \qquad \eta = \frac{r}{R}$$

then:

$$\nabla^2\theta - \frac{2h}{kb}\theta = 0$$

In cylindrical polar coordinates, for axisymmetrical cases and thin disks:

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{\eta}\frac{d\theta}{d\eta} - \frac{2hR^2}{kb}\theta = 0$$

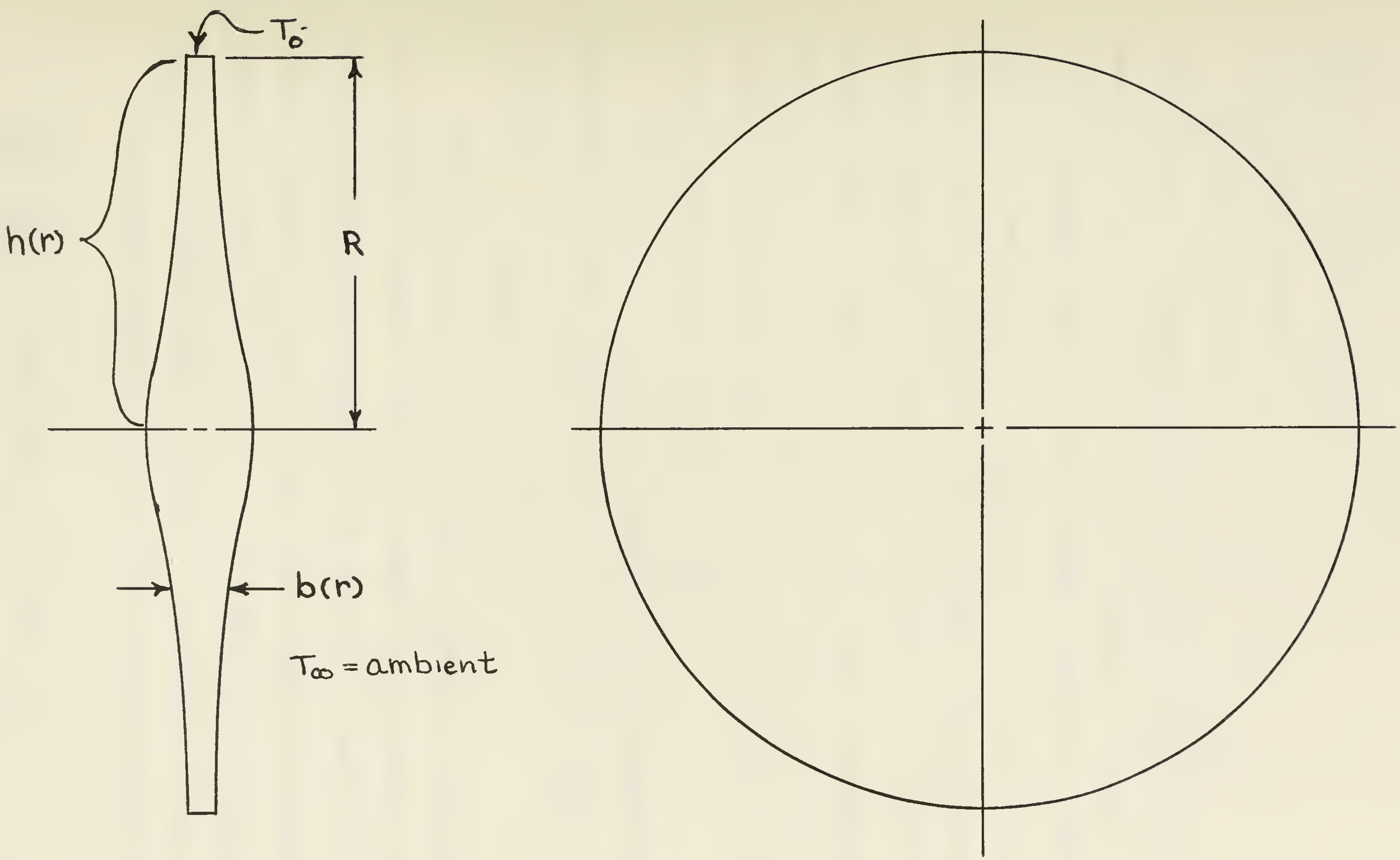


Figure 1: Sketch Illustrating Geometry and Boundary Conditions.

Define: $$\lambda(\eta) = \frac{2hR^2}{kb}$$

The boundary conditions of interest are:

$$\frac{d\theta}{d\eta} = 0 \quad \text{at} \quad \eta = 0 \quad ; \quad \theta = \theta_o \quad \text{at} \quad \eta = 1$$

Another set of boundary conditions would perhaps specify the heat flux at $\eta = 1$ however for two reasons this was not done in the present study, first the heat transfer coefficient is high for flow through a turbine as a result of high velocities hence the wall temperature will closely approximate the bulk fluid temperature; secondly it is difficult to determine the actual heat transfer coefficient for the flow through the turbine because of the complex geometry. It should be noted that the problem of specified heat flux at rotor tip can be handled in a way similar to the present problem.

The present analysis utilized a general second degree polynomial to approximate the distribution of $\lambda(\eta)$. This was written:

$$\lambda(\eta) = A\eta^2 + B\eta + C$$

This distribution was chosen because of the general ease with which a parabola may be fitted to a reasonably smooth set of data; certainly property data will show few discontinuities. Notice that both constant values and linear distributions are subsets of this polynomial. More complex polynomials may be used at a correspondent increase in algebraic labor.

The final differential equation to be solved takes the form:

$$-\frac{d}{d\eta}\left(\eta\frac{d\theta}{d\eta}\right) + (A\eta^2 + B\eta + C)\eta\theta = 0$$

I attempted to cast this equation into some form which would allow direct analytic solution through the various theorems available for Bessel type differential equations, none of the theorems used were applicable to this form. This is the differential equation which is solved by the approximate methods of the variational calculus.

Formulation of the Variational Problem.

Allow us to consider only the problem in which the approximate solution will satisfy the boundary conditions, we may then write the functional for a homogeneous differential equation of the second order as:

$$J(\theta) = \int_0^1 \left[ \eta (\theta')^2 + \lambda(\eta)\, \eta\, \theta^2 \right] d\eta$$

For treatment of more general cases which allow more freedom in the choice of the approximating function, the reader is referred to Collatz "Numerical Solutions to Differential Equations".

The Rayleigh Ritz method may be used to write the solution in the general form:

$$\theta(\eta) = \sum_{j=1}^{n} \alpha_j f_j(\eta)$$

Several specific polynomials were used in the selection of the most applicable one. These polynomials are listed in Table I and the resultant temperature distributions are shown by comparison in Figure 2.

The general approach to solution was to substitute the chosen approximation into the functional then carry out the indicated integration to obtain a function in terms of the

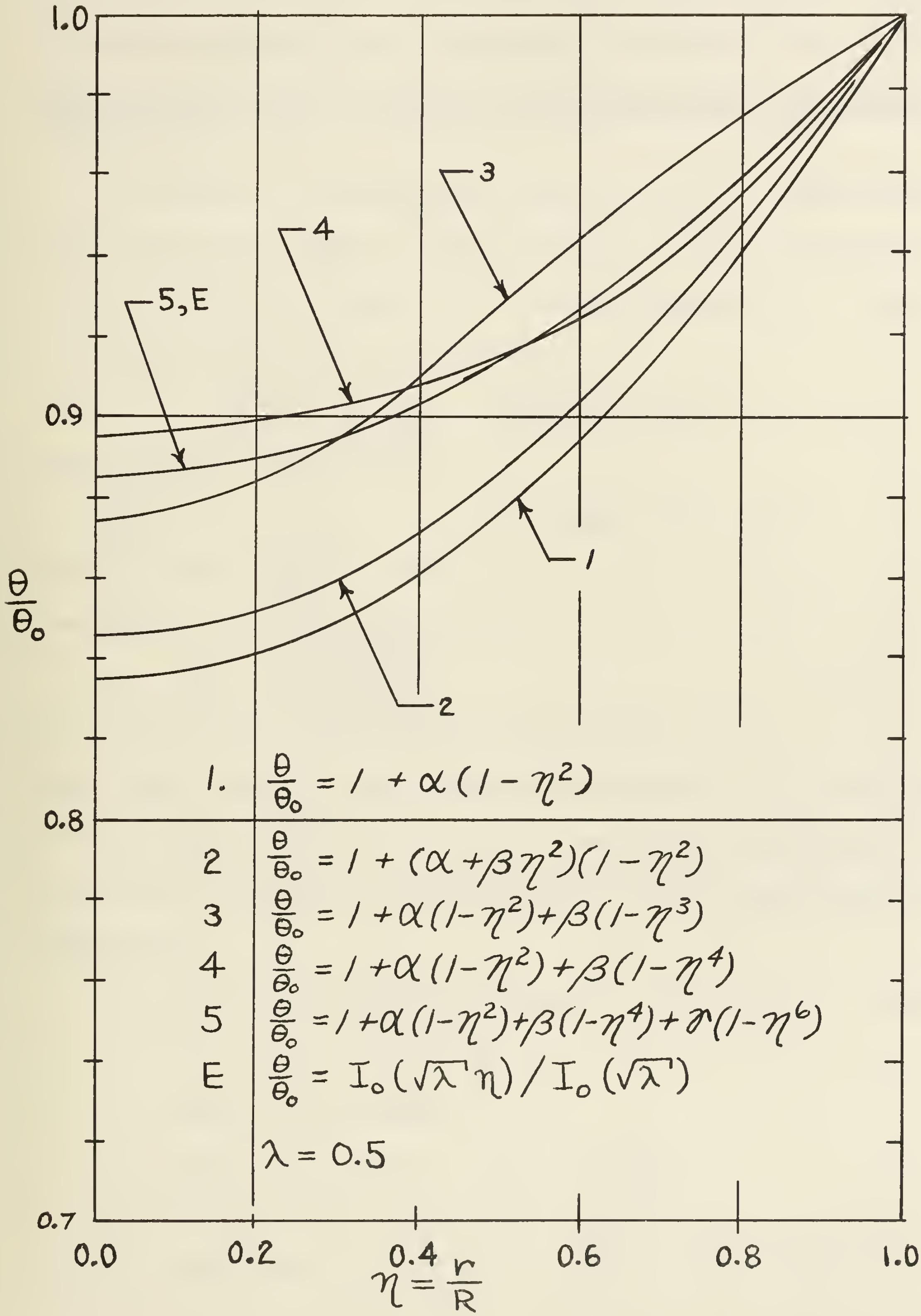


Figure 2: COMPARISON OF TEMPERATURE DISTRIBUTIONS USING VARIOUS APPROXIMATING POLYNOMIALS.

constants $\alpha_i$ . These constants were then treated as variables of the functional and the extremum obtained in the usual way.

## Exact Solution for a Constant Thickness Disk with Constant Properties.

A check of the general validity of the proposed approximate method was made by comparison to the exact solution of a constant thickness disk with constant properties. Recall the applicable differential equation:

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{\eta}\frac{d\theta}{d\eta} - \frac{2hR^2}{kb}\theta = 0$$

with boundary conditions:

$$\frac{d\theta}{d\eta} = 0 , \ \eta = 0 \qquad ; \qquad \theta = \theta_0 \ \text{at} \ \eta = 1$$

for the case $\frac{2hR^2}{kb}$ = const = $\lambda$

The simple solution results that:

$$\frac{\theta}{\theta_0} = \frac{I_0(\sqrt{\lambda}\,\eta)}{I_0(\sqrt{\lambda})}$$

$I_o$ = hyperbolic Bessel function of order zero

This exact solution was used as a criterion for the accuracy of the approximation. As will be seen the approximate technique reproduced this exact solution with no distinguishable error.

## Discussion

The various polynomials used are shown below table I.

1. $\theta = \theta_0[1 + \alpha(1-\eta^2)]$
2. $\theta = \theta_0[1 + (1-\eta^2)][\alpha + \beta\eta^2]$
3. $\theta = \theta_0[1 + \alpha(1-\eta^2) + \beta(1-\eta^3)]$
4. $\theta = \theta_0[1 + \alpha(1-\eta^2) + \beta(1-\eta^4)]$
5. $\theta = \theta_0[1 + \alpha(1-\eta^2) + \beta(1-\eta^4) + \gamma(1-\eta^6)]$

Notice that (1) is just the first approximation to all the other functions. By far the best agreement to the exact solution was obtained by number five. It is interesting to note that numbers 1 and 2 do not allow the function much freedom at the point = 0, while 3 and 4 allow $\alpha$ and $\beta$ to be arbitrary at $\eta$ = 0, and 5 allows $\alpha$, $\beta$, and $\gamma$ to be arbitrary at $\eta$ = 0. This seems to indicate that the greater the freedom given the coefficients the more rapidly will the solution approach the exact. Polynomial number 3 allows for an inversion in the temperature profile, a condition which is not physically reasonable for the particular problem being solved. See Figure 2 for comparison of the various approximation functions.

A detailed derivation for the case number 5 is presented in the appendix for the case of a parabolic distribution of the variable $\lambda(\eta)$. This was treated in the form:

$$\lambda(\eta) = A\eta^2 + B\eta + C$$

A, B. and C would be specified by the empirical fit of the experimental data for a second degree polynomial for the particular problem being studied. It would be expected that the distribution would be nearly linear for most practical cases and as a result the squared term coefficient would be small. An actual turbine might be expected to decrease in width as the radius increased, the thermal conductivity may either decrease or increase with radius (temperature) depending upon the material, and the heat transfer coefficient would be expected to

increase with radius due to the higher relative velocities. If the term $\lambda$ is examined it may be seen how these properties influence the parameter.

$$\lambda = \frac{2hR^2}{kb}$$

$\lambda$ tends to increase with radius and increased heat transfer coefficient, and increases with decreased thermal conductivity and thickness. A logical calculation procedure is to assume the properties constant for the first approximation then use the deduced temperature distribution for further approximations. The results of several computations for ranges of values of A, B, and C are presented in Table II. This table presents the coefficients for the expression for

$$\lambda(\eta) = A\eta^2 + B\eta + C$$

and also presents the coefficients $\alpha$, $\beta$, and $\gamma$ for the temperature distribution:

$$\theta/\theta_0 = 1 + \alpha(1-\eta^2) + \beta(1-\eta^4) + \gamma(1-\eta^6)$$

The computer program also calculates the temperature distribution but this vast printout would fill a book and the coefficients presented here allow for the same information.

Example of Table useage.

The use of the Table is completely straightforward. Suppose we fit a parabola to $\lambda$ which as the form:

$$\lambda = .01\eta^2 + .1\eta + 1$$

then we enter the table at the row where A = .01, B = .1, and C = 1. and read from the same row $\alpha$ = .19829, $\beta$ = .020819, and $\gamma$ = .000625. The temperature distribution is then:

$$\theta/\theta_0 = 1 + .19829(1-\eta^2) + .020819(1-\eta^4) + .000625(1-\eta^6)$$

Table 2 RESULTS OF ANALYSIS FOR TEMPERATURE DISTRIBUTIONS IN THIN CIRCULAR DISKS

| COEFFICIENTS OF LAMBDA | | | COEFFICIENTS FOR TEMPERATURE DISTRIBUTION | | |
|---|---|---|---|---|---|
| A | B | C | ALPHA | BETA | GAMMA |
| .000 | .000 | .100 | -.24387E-01 | -.15241E-03 | -.42363E-06 |
| .000 | .000 | .500 | -.11072E+00 | -.[illegible]E-02 | -.[illegible]22E-04 |
| .000 | .000 | 1.000 | -.19746E+00 | -.12332E-01 | -.35524E-03 |
| .000 | .000 | 2.000 | -.31930E+00 | -.39735E-01 | -.23803E-02 |
| .000 | .000 | 5.000 | -.46387E+00 | -.14175E+00 | -.23967E-0[illegible] |
| .000 | .000 | 10.000 | -.45552E+00 | -.25517E+00 | -.11023E+[illegible] |
| .000 | .001 | .100 | -.24421E-01 | -.24620E-03 | .19599E-04 |
| .000 | .001 | .500 | -.11074E+00 | -.35474E-02 | -.33150E-04 |
| .000 | .001 | 1.000 | -.19747E+00 | -.12413E-01 | -.34424E-03 |
| .000 | .001 | 2.000 | -.31929E+00 | -.39353E-01 | -.23773E-02 |
| .000 | .001 | 5.000 | -.46385E+00 | -.14179E+00 | -.[illegible]E-01 |
| .000 | .001 | 10.000 | -.45550E+00 | -.25517E+00 | -.11028E+00 |
| .000 | .010 | .100 | -.24733E-01 | -.10894E-0[illegible] | .19933E-03 |
| .000 | .010 | .500 | -.11094E+00 | -.43391E-02 | .[illegible]0632E-03 |
| .000 | .010 | 1.000 | -.19756E+00 | -.13141E-01 | -.24549E-03 |
| .000 | .010 | 2.000 | -.31924E+00 | -.40462E-01 | -.23504E-02 |
| .000 | .010 | 5.000 | -.46366E+00 | -.14211E+00 | -.24099E-01 |
| .000 | .010 | 10.000 | -.45534E+00 | -.25519E+00 | -.11053E+00 |
| .000 | .100 | .100 | -.27785E-01 | -.94297E-02 | .19501E-02 |
| .000 | .100 | .500 | -.11285E+00 | -.12171E-01 | .14018E-02 |
| .000 | .100 | 1.000 | -.19640E+00 | -.20350E-01 | .70613E-03 |
| .000 | .100 | 2.000 | -.31870E+00 | -.46494E-01 | -.21069E-02 |
| .000 | .100 | 5.000 | -.46177E+00 | -.14529E+00 | -.25293E-01 |
| .000 | .100 | 10.000 | -.45370E+00 | -.25539E+00 | -.11295E+00 |
| .000 | 1.000 | .100 | -.55240E-01 | -.84301E-01 | .15136E-[illegible] |
| .000 | 1.000 | .500 | -.13000E+00 | -.82654E-01 | .11358E-01 |
| .000 | 1.000 | 1.000 | -.20573E+00 | -.85358E-01 | .68672E-02 |
| .000 | 1.000 | 2.000 | -.31330E+00 | -.10098E+00 | -.22984E-02 |
| .000 | 1.000 | 5.000 | -.44385E+00 | -.17366E+00 | -.38604E-01 |
| .000 | 1.000 | 10.000 | -.43813E+00 | -.25564E+00 | -.13773E+00 |
| .000 | 10.000 | .100 | -.17617E+00 | -.33543E+00 | -.94207E-01 |
| .000 | 10.000 | .500 | -.20270E+00 | -.31934E+00 | -.10964E+00 |
| .000 | 10.000 | 1.000 | -.23089E+00 | -.30185E+00 | -.12316E+00 |
| .000 | 10.000 | 2.000 | -.27392E+00 | -.27341E+00 | -.16330E+00 |
| .000 | 10.000 | 5.000 | -.33631E+00 | -.2146[illegible]E+00 | -.26143E+00 |
| .000 | 10.000 | 10.000 | -.34830E+00 | -.12966E+00 | -.42293E+00 |
| .001 | .000 | .100 | -.24385E-01 | -.21336E-03 | -.12826E-05 |
| .001 | .000 | .500 | -.11071E+00 | -.35144E-02 | -.52955E-0[illegible] |
| .001 | .000 | 1.000 | -.19745E+00 | -.12360E-01 | -.36276E-[illegible] |
| .001 | .000 | 2.000 | -.31928E+00 | -.39821E-01 | -.2393[illegible] |
| .001 | .000 | 5.000 | -.46385E+00 | -.14176E+00 | -.23992E-01 |
| .001 | .000 | 10.000 | -.45551E+00 | -.25515E+00 | -.11029E+00 |
| .001 | .001 | .100 | -.24419E-01 | -.30713E-03 | .18733E-04 |
| .001 | .001 | .500 | -.11074E+00 | -.36024E-02 | -.37194E-04 |
| .001 | .001 | 1.000 | -.19746E+00 | -.12461E-01 | -.35179E-03 |
| .001 | .001 | 2.000 | -.31928E+00 | -.39889E-01 | -.23906E-02 |
| .001 | .001 | 5.000 | -.46383E+00 | -.14180E+00 | -.24006E-01 |
| .001 | .001 | 10.000 | -.[illegible]E+00 | -.25519E+00 | -.110[illegible]E+0[illegible] |
| .001 | .010 | .100 | -.24728E-01 | -.11502E-02 | .[illegible]E-03 |
| .001 | .010 | .500 | -.11093E+00 | -.43939E-02 | .[illegible]22E-03 |
| .001 | .010 | 1.000 | -.19755E+00 | -.13189E-01 | -.25309E-03 |
| .001 | .010 | 2.000 | -.31922E+00 | -.40493E-01 | -.23537E-02 |
| .001 | .010 | 5.000 | -.46364E+00 | -.14212E+00 | -.24123E-01 |
| .001 | .010 | 10.000 | -.45533E+00 | -.25517E+00 | -.11056E+00 |

| | | | | | |
|---|---|---|---|---|---|
| .001 | .100 | .100 | -.27783E-01 | -.94891E-02 | .1948[illegible]E-02 |
| .001 | .100 | .500 | -.11285E+00 | -.12225E-01 | .1477[illegible]-02 |
| .001 | .100 | 1.000 | -.19339E+00 | -.20397E-01 | .69604E-03 |
| .001 | .100 | 2.000 | -.31868E+00 | -.46529E-01 | -.21226E-02 |
| .001 | .100 | 5.000 | -.46176E+00 | -.14530E+00 | -.25318E-01 |
| .001 | .100 | 10.000 | -.45369E+00 | -.25537E+00 | -.11298E+00 |
| .001 | 1.000 | .100 | -.55237E-01 | -.84348E-01 | .15129E-01 |
| .001 | 1.000 | .500 | -.12999E+00 | -.82696E-01 | .11358E-01 |
| .001 | 1.000 | 1.000 | -.20572E+00 | -.85395E-01 | .68549E-02 |
| .001 | 1.000 | 2.000 | -.31329E+00 | -.10101E+00 | -.23153E-02 |
| .001 | 1.000 | 5.000 | -.44383E+00 | -.17367E+00 | -.38630E-01 |
| .001 | 1.000 | 10.000 | -.43817E+00 | -.25562E+00 | -.13776E+00 |
| .001 | 10.000 | .100 | -.17617E+00 | -.33542E+00 | -.9423[illegible]E-01 |
| .001 | 10.000 | .500 | -.20270E+00 | -.31933E+00 | -.10967E+00 |
| .001 | 10.000 | 1.000 | -.23089E+00 | -.30183E+00 | -.12819E+00 |
| .001 | 10.000 | 2.000 | -.27391E+00 | -.27339E+00 | -.16334E+00 |
| .001 | 10.000 | 5.000 | -.33631E+00 | -.21482E+00 | -.26147E+00 |
| .001 | 10.000 | 10.000 | -.34830E+00 | -.12963E+00 | -.42296E+00 |
| .010 | .000 | .100 | -.24372E-01 | -.76138E-03 | -.91932E-05 |
| .010 | .000 | .500 | -.11066E+00 | -.40090E-02 | -.89482E-04 |
| .010 | .000 | 1.000 | -.19735E+00 | -.12813E-01 | -.43085E-03 |
| .010 | .000 | 2.000 | -.31914E+00 | -.40145E-01 | -.25137E-02 |
| .010 | .000 | 5.000 | -.46369E+00 | -.14185E+00 | -.24213E-01 |
| .010 | .000 | 10.000 | -.45541E+00 | -.25501E+00 | -.11059E+00 |
| .010 | .001 | .100 | -.24406E-01 | -.85501E-03 | .107[illegible]E-04 |
| .010 | .001 | .500 | -.11068E+00 | -.40969E-02 | -.73781E-04 |
| .010 | .001 | 1.000 | -.19736E+00 | -.12894E-[illegible] | -.41989E-03 |
| .010 | .001 | 2.000 | -.31913E+00 | -.40213E-01 | -.25107E-02 |
| .010 | .001 | 5.000 | -.46367E+00 | -.14108E+00 | -.24226E-01 |
| .010 | .001 | 10.000 | -.45539E+00 | -.25501E+00 | -.11062E+00 |
| .010 | .010 | .100 | -.24715E-01 | -.16968E-02 | .16988E-03 |
| .010 | .010 | .500 | -.11087E+00 | -.48873E-02 | .67156E-04 |
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| .010 | .010 | 2.000 | -.31908E+00 | -.40821E-01 | -.24642E-02 |
| .010 | .010 | 5.000 | -.46348E+00 | -.14220E+00 | -.24344E-01 |
| .010 | .010 | 10.000 | -.45522E+00 | -.25504E+00 | -.11066E+00 |
| .010 | .100 | .100 | -.27768E-01 | -.10023E-01 | .19345E-02 |
| .010 | .100 | .500 | -.11279E+00 | -.12707E-01 | .14352E-[illegible] |
| .010 | .100 | 1.000 | -.19829E+00 | -.20819E-01 | .62533E-03 |
| .010 | .100 | 2.000 | -.31854E+00 | -.46844E-01 | -.22463E-02 |
| .010 | .100 | 5.000 | -.461[illegible]E+00 | -.14538E+00 | -.25540E-01 |
| .010 | .100 | 10.000 | -.45359E+00 | -.25523E+00 | -.11328E+00 |
| .010 | 1.000 | .100 | -.55214E-01 | -.84770E-01 | .15066E-01 |
| .010 | 1.000 | .500 | -.12993E+00 | -.83077E-01 | .11272E-01 |
| .010 | 1.000 | 1.000 | -.20563E+00 | -.85725E-01 | .67442E-02 |
| .010 | 1.000 | 2.000 | -.31316E+00 | -.10125E+00 | -.24682E-02 |
| .010 | 1.000 | 5.000 | -.44369E+00 | -.17370E+00 | -.38867E-01 |
| .010 | 1.000 | 10.000 | -.43808E+00 | -.25546E+00 | -.13807E+00 |
| .010 | 10.000 | .100 | -.17616E+00 | -.33531E+00 | -.94515E-01 |
| .010 | 10.000 | .500 | -.20268E+00 | -.31921E+00 | -.10995E+00 |
| .010 | 10.000 | 1.000 | -.23086E+00 | -.30170E+00 | -.12848E+00 |
| .010 | 10.000 | 2.000 | -.27388E+00 | -.27324E+00 | -.16363E+00 |
| .010 | 10.000 | 5.000 | -.33628E+00 | -.21460E+00 | -.26178E+00 |
| .010 | 10.000 | 10.000 | -.34830E+00 | -.12933E+00 | -.42529E+00 |
| .100 | .000 | .100 | -.24241E-01 | -.61934E-02 | -.1074[illegible] |
| .100 | .000 | .500 | -.11008E+00 | -.89114E-0[illegible] | -.471[illegible] |
| .100 | .000 | 1.000 | -.19637E+00 | -.17102E-01 | [illegible] |
| .100 | .000 | 2.000 | -.31770E+00 | -.43354E-01 | -.3725[illegible]E-02 |
| .100 | .000 | 5.000 | -.46208E+00 | -.14269E+00 | -.26[illegible]E-01 |
| .100 | .000 | 10.000 | -.45441E+00 | -.25361E+00 | -.11[illegible]59E+00 |
| .100 | .001 | .100 | -.24275E-01 | -.62856E-02 | -.[illegible]E-04 |
| .100 | .001 | .500 | -.11010E+00 | -.89981E-02 | -.45331E-03 |

| | | | | | |
|---|---|---|---|---|---|
| .100 | .001 | 1.000 | -.19638E+00 | -.17182E-01 | -.11153E-02 |
| .100 | .001 | 2.000 | -.31769E+00 | -.43420E-01 | -.37226E-02 |
| .100 | .001 | 5.000 | -.46206E+00 | -.14273E+00 | -.26437E-01 |
| .100 | .001 | 10.000 | -.45439E+00 | -.25361E+00 | -.11361E+00 |
| .100 | .010 | .100 | -.24583E-01 | -.71150E-02 | .85506E-04 |
| .100 | .010 | .500 | -.11029E+00 | -.97772E-02 | -.32025E-03 |
| .100 | .010 | 1.000 | -.19647E+00 | -.17899E-01 | -.10213E-02 |
| .100 | .010 | 2.000 | -.31764E+00 | -.44021E-01 | -.36993E-02 |
| .100 | .010 | 5.000 | -.46187E+00 | -.14304E+00 | -.26557E-01 |
| .100 | .010 | 10.000 | -.45423E+00 | -.25363E+00 | -.11386E+00 |
| .100 | .100 | .100 | -.27626E-01 | -.15319E-01 | .17755E-02 |
| .100 | .100 | .500 | -.11221E+00 | -.17486E-01 | .99955E-03 |
| .100 | .100 | 1.000 | -.19732E+00 | -.24998E-01 | -.11600E-03 |
| .100 | .100 | 2.000 | -.31711E+00 | -.49963E-01 | -.34931E-02 |
| .100 | .100 | 5.000 | -.46001E+00 | -.14618E+00 | -.27770E-01 |
| .100 | .100 | 10.000 | -.45260E+00 | -.25380E+00 | -.11629E+00 |
| .100 | 1.000 | .100 | -.54991E-01 | -.88956E-01 | .14416E-01 |
| .100 | 1.000 | .500 | -.12935E+00 | -.86842E-01 | .10402E-01 |
| .100 | 1.000 | 1.000 | -.20471E+00 | -.88997E-01 | .56250E-02 |
| .100 | 1.000 | 2.000 | -.31187E+00 | -.10363E+00 | -.40053E-02 |
| .100 | 1.000 | 5.000 | -.44227E+00 | -.17407E+00 | -.41241E-01 |
| .100 | 1.000 | 10.000 | -.43721E+00 | -.25383E+00 | -.14112E+00 |
| .100 | 10.000 | .100 | -.17606E+00 | -.33418E+00 | -.97287E-01 |
| .100 | 10.000 | .500 | -.20251E+00 | -.31799E+00 | -.11276E+00 |
| .100 | 10.000 | 1.000 | -.23062E+00 | -.30037E+00 | -.13132E+00 |
| .100 | 10.000 | 2.000 | -.27355E+00 | -.27169E+00 | -.16656E+00 |
| .100 | 10.000 | 5.000 | -.33596E+00 | -.21243E+00 | -.26490E+00 |
| .100 | 10.000 | 10.000 | -.34827E+00 | -.12640E+00 | -.42659E+00 |
| 1.000 | .000 | .100 | -.23378E-01 | -.55900E-01 | -.29156E-02 |
| 1.000 | .000 | .500 | -.10490E+00 | -.53764E-01 | -.58966E-02 |
| 1.000 | .000 | 1.000 | -.18731E+00 | -.56292E-01 | -.94523E-02 |
| 1.000 | .000 | 2.000 | -.30418E+00 | -.72479E-01 | -.16872E-01 |
| 1.000 | .000 | 5.000 | -.44677E+00 | -.14947E+00 | -.49011E-01 |
| 1.000 | .000 | 10.000 | -.44492E+00 | -.23884E+00 | -.14368E+00 |
| 1.000 | .001 | .100 | -.23411E-01 | -.55979E-01 | -.29020E-02 |
| 1.000 | .001 | .500 | -.10492E+00 | -.53838E-01 | -.58865E-02 |
| 1.000 | .001 | 1.000 | -.18732E+00 | -.56362E-01 | -.94462E-02 |
| 1.000 | .001 | 2.000 | -.30417E+00 | -.72537E-01 | -.16873E-01 |
| 1.000 | .001 | 5.000 | -.44676E+00 | -.14950E+00 | -.49026E-01 |
| 1.000 | .001 | 10.000 | -.44490E+00 | -.23884E+00 | -.14371E+00 |
| 1.000 | .010 | .100 | -.23708E-01 | -.56693E-01 | -.27800E-02 |
| 1.000 | .010 | .500 | -.10511E+00 | -.54512E-01 | -.57961E-02 |
| 1.000 | .010 | 1.000 | -.18741E+00 | -.56985E-01 | -.93916E-02 |
| 1.000 | .010 | 2.000 | -.30413E+00 | -.73061E-01 | -.16880E-01 |
| 1.000 | .010 | 5.000 | -.44658E+00 | -.14977E+00 | -.49161E-01 |
| 1.000 | .010 | 10.000 | -.44475E+00 | -.23884E+00 | -.14395E+00 |
| 1.000 | .100 | .100 | -.26650E-01 | -.63753E-01 | -.15997E-02 |
| 1.000 | .100 | .500 | -.10701E+00 | -.61179E-01 | -.49280E-02 |
| 1.000 | .100 | 1.000 | -.18831E+00 | -.63151E-01 | -.88777E-02 |
| 1.000 | .100 | 2.000 | -.30374E+00 | -.78248E-01 | -.16973E-01 |
| 1.000 | .100 | 5.000 | -.44488E+00 | -.15247E+00 | -.50522E-01 |
| 1.000 | .100 | 10.000 | -.44325E+00 | -.23881E+00 | -.14644E+00 |
| 1.000 | 1.000 | .100 | -.53228E-01 | -.12699E+00 | .64575E-02 |
| 1.000 | 1.000 | .500 | -.12409E+00 | -.12101E+00 | .39183E-03 |
| 1.000 | 1.000 | 1.000 | -.19628E+00 | -.11859E+00 | -.66919E-02 |
| 1.000 | 1.000 | 2.000 | -.29975E+00 | -.12492E+00 | -.20231E-01 |
| 1.000 | 1.000 | 5.000 | -.42874E+00 | -.17624E+00 | -.65377E-01 |
| 1.000 | 1.000 | 10.000 | -.42901E+00 | -.23686E+00 | -.17175E+00 |
| 1.000 | 10.000 | .100 | -.17543E+00 | -.32184E+00 | -.12527E+00 |
| 1.000 | 10.000 | .500 | -.20114E+00 | -.30485E+00 | -.14106E+00 |
| 1.000 | 10.000 | 1.000 | -.22855E+00 | -.28619E+00 | -.16002E+00 |
| 1.000 | 10.000 | 2.000 | -.27068E+00 | -.25539E+00 | -.19599E+00 |

| | | | | | |
|---|---|---|---|---|---|
| 1.000 | 10.000 | 5.000 | -.33317E+00 | -.19020E+00 | -.29612E+00 |
| 1.000 | 10.000 | 10.000 | -.34820E+00 | -.96837E-01 | -.45947E+00 |
| 10.000 | .000 | .100 | -.47405E-01 | -.23511E+00 | -.15296E+00 |
| 10.000 | .000 | .500 | -.98951E-01 | -.21034E+00 | -.16885E+00 |
| 10.000 | .000 | 1.000 | -.15326E+00 | -.18481E+00 | -.18729E+00 |
| 10.000 | .000 | 2.000 | -.23557E+00 | -.14715E+00 | -.22079E+00 |
| 10.000 | .000 | 5.000 | -.35688E+00 | -.89189E-01 | -.30881E+00 |
| 10.000 | .000 | 10.000 | -.39249E+00 | -.31690E-01 | -.45201E+00 |
| 10.000 | .001 | .100 | -.47430E-01 | -.23511E+00 | -.15298E+00 |
| 10.000 | .001 | .500 | -.98970E-01 | -.21035E+00 | -.16887E+00 |
| 10.000 | .001 | 1.000 | -.15328E+00 | -.18482E+00 | -.18731E+00 |
| 10.000 | .001 | 2.000 | -.23557E+00 | -.14716E+00 | -.22081E+00 |
| 10.000 | .001 | 5.000 | -.35688E+00 | -.89186E-01 | -.30883E+00 |
| 10.000 | .001 | 10.000 | -.39249E+00 | -.31673E-01 | -.45204E+00 |
| 10.000 | .010 | .100 | -.47654E-01 | -.23518E+00 | -.15315E+00 |
| 10.000 | .010 | .500 | -.99140E-01 | -.21042E+00 | -.16904E+00 |
| 10.000 | .010 | 1.000 | -.15339E+00 | -.18488E+00 | -.18748E+00 |
| 10.000 | .010 | 2.000 | -.23561E+00 | -.14721E+00 | -.22100E+00 |
| 10.000 | .010 | 5.000 | -.35682E+00 | -.89158E-01 | -.30907E+00 |
| 10.000 | .010 | 10.000 | -.39243E+00 | -.31519E-01 | -.45232E+00 |
| 10.000 | .100 | .100 | -.49878E-01 | -.23576E+00 | -.15479E+00 |
| 10.000 | .100 | .500 | -.10083E+00 | -.21103E+00 | -.17073E+00 |
| 10.000 | .100 | 1.000 | -.15455E+00 | -.18548E+00 | -.18925E+00 |
| 10.000 | .100 | 2.000 | -.23601E+00 | -.14767E+00 | -.22291E+00 |
| 10.000 | .100 | 5.000 | -.35629E+00 | -.88864E-01 | -.31139E+00 |
| 10.000 | .100 | 10.000 | -.39183E+00 | -.29971E-01 | -.45514E+00 |
| 10.000 | 1.000 | .100 | -.70669E-01 | -.23929E+00 | -.17246E+00 |
| 10.000 | 1.000 | .500 | -.11669E+00 | -.21487E+00 | -.18881E+00 |
| 10.000 | 1.000 | 1.000 | -.16542E+00 | -.18923E+00 | -.20792E+00 |
| 10.000 | 1.000 | 2.000 | -.23980E+00 | -.15020E+00 | -.24288E+00 |
| 10.000 | 1.000 | 5.000 | -.35129E+00 | -.84374E-01 | -.33521E+00 |
| 10.000 | 1.000 | 10.000 | -.38634E+00 | -.13586E-01 | -.48353E+00 |
| 10.000 | 10.000 | .100 | -.19711E+00 | -.12001E+00 | -.42299E+00 |
| 10.000 | 10.000 | .500 | -.21684E+00 | -.98633E-01 | -.44031E+00 |
| 10.000 | 10.000 | 1.000 | -.23847E+00 | -.74023E-01 | -.46123E+00 |
| 10.000 | 10.000 | 2.000 | -.27338E+00 | -.30394E-01 | -.50112E+00 |
| 10.000 | 10.000 | 5.000 | -.33404E+00 | .73913E-01 | -.61142E+00 |
| 10.000 | 10.000 | 10.000 | -.36842E+00 | .21931E+00 | -.78408E+00 |

TIME, 0 MINUTES AND 51 SECONDS

## Appendix A - Details of Computations

The functional may be written:

$$J(\theta) = \int_0^1 \left\{ \eta(\theta')^2 + A\eta^3\theta^2 + B\eta^2\theta^2 + C\eta\,\theta^2 \right\} d\eta$$

Define $\psi$ as follows:

$$\psi \equiv \frac{\theta}{\theta_0} = 1 + \alpha(1-\eta^2) + \beta(1-\eta^4) + \gamma(1-\eta^6)$$

The derivative becomes:

$$\psi' = -2\alpha\eta - 4\beta\eta^3 - 6\gamma\eta^5$$

Squaring

$$(\psi')^2 = 4\alpha^2\eta^2 + 16\alpha\beta\eta^4 + (24\alpha\gamma + 16\beta^2)\eta^6 + 48\beta\gamma\eta^8 + 36\gamma^2\eta^{10}$$

Squaring the function.

$$\begin{aligned}\psi^2 = {} & 1 + 2\alpha + 2\beta + 2\gamma + \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma \\ & + 2\alpha\gamma - (2\alpha + 2\alpha^2 + 2\alpha\beta + 2\alpha\gamma)\eta^2 + \\ & (\alpha^2 - 2\beta - 2\beta^2 - 2\alpha\beta - 2\beta\gamma)\eta^4 + \\ & (2\alpha\beta - 2\gamma - 2\gamma^2 - 2\beta\gamma - 2\alpha\gamma)\eta^6 + \\ & (\beta^2 + 2\alpha\gamma)\eta^8 + 2\beta\gamma\eta^{10} + \gamma^2\eta^{12}\end{aligned}$$

Define for ease of algebra.

$$K_1 \equiv 1 + 2\alpha + 2\beta + 2\gamma + \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma$$

$$K_2 \equiv -(2\alpha + 2\alpha^2 + 2\alpha\beta + 2\alpha\gamma)$$

$$K_3 \equiv \alpha^2 - 2\beta - 2\beta^2 - 2\alpha\beta - 2\beta\gamma$$

$$K_4 \equiv 2\alpha\beta - 2\gamma - 2\gamma^2 - 2\beta\gamma - 2\alpha\gamma$$

$$K_5 \equiv \beta^2 + 2\alpha\gamma$$

$$K_6 \equiv 2\beta\gamma$$

$$K_7 \equiv \gamma^2$$

Integrals involving $\psi^2$ may be evaluated:

$$\int_0^1 \eta^3 \psi^2 d\eta = \tfrac{1}{4}K_1 + \tfrac{1}{6}K_2 + \tfrac{1}{8}K_3 + \tfrac{1}{10}K_4 + \tfrac{1}{12}K_5 + \tfrac{1}{14}K_6 + \tfrac{1}{16}K_7$$

$$\int_0^1 \eta^2 \psi^2 d\eta = \tfrac{1}{3}K_1 + \tfrac{1}{5}K_2 + \tfrac{1}{7}K_3 + \tfrac{1}{9}K_4 + \tfrac{1}{11}K_5 + \tfrac{1}{13}K_6 + \tfrac{1}{15}K_7$$

$$\int_0^1 \eta \psi^2 d\eta = \tfrac{1}{2}K_1 + \tfrac{1}{4}K_2 + \tfrac{1}{6}K_3 + \tfrac{1}{8}K_4 + \tfrac{1}{10}K_5 + \tfrac{1}{12}K_6 + \tfrac{1}{14}K_7$$

The integral involving $(\psi')^2$ becomes:

$$\int_0^1 \eta(\psi')^2 d\eta = \alpha^2 + \tfrac{8}{3}\alpha\beta + 3\alpha\gamma + 2\beta^2 + \tfrac{24}{5}\beta\gamma + 3\gamma^2$$

Substituting these integrals the functional may be written:

$$\begin{aligned} J(\psi) = {} & \alpha^2 + \tfrac{8}{3}\alpha\beta + 3\alpha\gamma + 2\beta^2 + \tfrac{24}{5}\beta\gamma + 3\gamma^2 \\ & + (\tfrac{1}{4}A + \tfrac{1}{3}B + \tfrac{1}{2}C)K_1 + (\tfrac{1}{6}A + \tfrac{1}{5}B + \tfrac{1}{4}C)K_2 \\ & + (\tfrac{1}{8}A + \tfrac{1}{7}B + \tfrac{1}{6}C)K_3 + (\tfrac{1}{10}A + \tfrac{1}{9}B + \tfrac{1}{8}C)K_4 \\ & + (\tfrac{1}{12}A + \tfrac{1}{11}B + \tfrac{1}{10}C)K_5 + (\tfrac{1}{14}A + \tfrac{1}{13}B + \tfrac{1}{12}C)K_6 \\ & + (\tfrac{1}{16}A + \tfrac{1}{15}B + \tfrac{1}{14}C)K_7 \end{aligned}$$

The functional is now differentiated with respect to the coefficients $\alpha$, $\beta$, and $\gamma$ to obtain the extremal. Define for convenience:

$$L_1 \equiv \tfrac{1}{4}A + \tfrac{1}{3}B + \tfrac{1}{2}C \qquad L_2 \equiv \tfrac{1}{6}A + \tfrac{1}{5}B + \tfrac{1}{4}C$$

$$L_3 \equiv \tfrac{1}{8}A + \tfrac{1}{7}B + \tfrac{1}{6}C \qquad L_4 \equiv \tfrac{1}{10}A + \tfrac{1}{9}B + \tfrac{1}{8}C$$

$$L_5 \equiv \tfrac{1}{12}A + \tfrac{1}{11}B + \tfrac{1}{10}C \qquad L_6 \equiv \tfrac{1}{14}A + \tfrac{1}{13}B + \tfrac{1}{12}C$$

$$L_7 \equiv \tfrac{1}{16}A + \tfrac{1}{15}B + \tfrac{1}{14}C$$

Differentiating with $\alpha$;

$$\begin{aligned} \frac{\partial J}{\partial \alpha} = {} & 2(1 + L_1 - 2L_2 + L_3)\alpha + 2(\tfrac{4}{3} + L_1 - L_2 - L_3 + L_4)\beta \\ & + (3 + 2L_1 - 2L_2 - 2L_4 + 2L_5)\gamma + (2L_1 - 2L_2) \end{aligned}$$

defining the coefficients of $\alpha$, $\beta$, and $\gamma$ by $M_1'$; $M_2'$ and $M_3'$.
The derivative may be written:

$$\frac{\partial J}{\partial \alpha} = M_1'\alpha + M_2'\beta + M_3'\gamma + M_4' = 0$$

differentiation with respect to $\beta$ yields:

$$\frac{\partial J}{\partial \beta} = \left(\frac{8}{3} + 2L_1 - 2L_2 - 2L_3 + 2L_4\right)\alpha + \left(4 + 2L_1 - 4L_3 + 2L_5\right)\beta + \left(\frac{24}{5} + 2L_1 - 2L_3 - 2L_4 + 2L_6\right)\gamma + (2L_1 - 2L_3)$$

Defining $M_1''$, $M_2''$, $M_3''$ in a similar way:

$$\frac{\partial J}{\partial \beta} = M_1''\alpha + M_2''\beta + M_3''\gamma + M_4'' = 0$$

finally differentiation with respect to $\gamma$ yields:

$$\frac{\partial J}{\partial \gamma} = (3 + 2L_1 - 2L_2 - 2L_4 + 2L_5)\alpha + \left(\frac{24}{5} + 2L_1 - 2L_3 - 2L_4 + 2L_6\right)\beta + (6 + 2L_1 - 4L_4 + 2L_7)\gamma + (2L_1 - 2L_4) = 0$$

Define $M_1''$, $M_2''$, and $M_3''$ in a similar way

$$\frac{\partial J}{\partial \gamma} = M_1'''\alpha + M_2'''\beta + M_3'''\gamma + M_4''' = 0$$

The system of linear equations to be solved for $\alpha$, $\beta$, and $\gamma$ becomes:

$$M_1'\alpha + M_2'\beta + M_3'\gamma = -M_4'$$
$$M_1''\alpha + M_2''\beta + M_3''\gamma = -M_4''$$
$$M_1'''\alpha + M_2'''\beta + M_3'''\gamma = -M_4'''$$

Once the desired values for A, B, and C are known the included computer program will evaluate $\alpha$, $\beta$, and $\gamma$ according to the above method (see Appendix B).

## APPENDIX B

### Computer Program for Evaluation of Temperature Profiles

Program TEMP carries out the computations outlined in Appendix A. This program is written in Fortran 60 for use on a Control Data Corporation 1604 digital computer.

Coefficients used to describe the parameter $\lambda$ in terms of a second degree polynomial are given the program as input.

$$\lambda(\eta) = A\eta^2 + B\eta + C$$

Output information consists of the temperature distributions as functions of the non-dimensional radius. The coefficients of the temperature distribution polynomial are also printed out.

A fixed point variable, N, determines the number of sets of coefficients to be computed. Symbols and formats for the input data are as follows:

| Variable | Fortran | Format |
|---|---|---|
| A | A | 8F10.5 |
| B | B | 8F10.5 |
| C | D | 8F10.5 |
| | N | I10 |

A sample printout of the program and typical data are presented in Table 3. A sample output data sheet is presented in Table 2.

Table 3 SAMPLE PROGRAM PRINTOUT

[illegible]

```
 20 FORMAT(3F10.3,3X,3(1E11.5,3X))
    END
    SUBROUTINE PARAFIT (C,B,A)
C   THIS SUBROUTINE WILL FIT A PARABOLA TO THREE POINTS
    DIMENSION C(3,3), B(3), CM(3,3), A(3), DETCM(3), K(3)
    DETC=(C(1,1)*C(2,2)*C(3,3)+C(1,2)*C(2,3)*C(3,1)+
   1C(1,3)*C(2,1)*C(3,2)) - (C(1,1)*C(2,3)*C(3,2)+
   2C(1,2)*C(2,1)*C(3,3)+C(1,3)*C(2,2)*C(3,1))
    DO 900 K=1,3
    K=K
    DO 905 N=1,3
    DO 905 J= 1,3
905 CM(N,J) = C(N,J)
    DO 904  I=1,3
904 CM(I,K) - B(I)
    DETCM(K) = (CM(1,1)*CM(2,2)*CM(3,3) +CM(1,2)*CM(2,3)*CM(3,1)+
   1CM(1,3)*CM(2,1)*CM(3,2))-(CM(1,1)*CM(2,3)*CM(3,2)+
   2CM(1,2)*CM(2,1)*CM(3,3)+CM(1,3)*CM(2,2).*CM(3,1))
900 A(K) = DETCM(K)/DETC
    RETURN
    END
    END

       6
.000      0.001     0.01      0.1       1.0       10.0
.000      0.001     0.01      0.1       1.0       10.0
.10       0.50      1.0       2.0       5.0       10.0
```